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A mathematical programming analysis of public transit systems

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Abstract

We illustrate how mathematical programming models can be used to analyse public transit systems. In particular, we analyse the location of bus garages and the allocation of vehicles to these facilities for the Vancouver (Canada) Regional Transit System. The model considers, among other items, vehicle deadhead cost and the capital cost of constructing new garages. Besides the ability to determine cost-minimizing solutions, we explore the use of mathematical programming to examine the effects of different scenarios. © 2002 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Our existence in time is determined for us, but we are largely free to select our location. Losch [1, p. 3]

Public transit systems, by virtue of their network design and substantial costs, would appear to represent a fruitful area for mathematical modelling. In this paper, we explore the development of a mixed integer programming (MIP) optimization model to determine the best number, location and size of transit centres to serve an existing (or planned) network of transit routes. Without careful analysis and review, the costs of providing bus service from inefficient transit centre locations can become quite enormous. These transit centres (also known as “bus garages” or “depots”) serve as facilities where buses are housed and various maintenance activities performed.

All buses begin their service day from an assigned transit centre and return at the conclusion of the day to that depot. Undoubtedly, the largest cost associated with bus garage location involves the “deadheading” of buses to their assigned routes. Generally, buses do not begin revenue service from the moment of departure from their transit centre. Likewise,

buses usually do not undertake revenue service to the bus garage at the end of the service period. A certain amount of time is required to travel “not-in-service” between the depot and route (initiation of service) and the route and depot (termination of service). This time is referred to as “deadhead” (nonrevenue transportation time).

A further cost impacting the location-allocation decisions of transit authorities concerns the capital cost of constructing new bus garages. These facilities incur substantial expense and, given the increased traffic load associated with their construction, are normally the subject of considerable public debate.

The development of a mathematical model to assist in transit depot location decisions is explored for the buses and route network of the Vancouver (Canada) Regional Transit System (VRTS), owned and operated by British Columbia (BC) Transit. We shall use our model to generate an optimal location-allocation decision. In addition, we shall explore the manner in which locations, allocations and overall costs are affected by different transit planning scenarios.

The format of this paper is as follows. Section 2 offers a review of the literature pertinent to this paper, while Section 3 discusses the MIP model used to solve the transit centre location problem. Section 4 describes the different transit planning scenarios explored with the MIP model. Conclusions are offered in Section 5.

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2. Literature review

The determination of the optimal number, size and location of “facilities” to serve a base of “customers” is one of a class of problems known as *location/allocation problems*. Research in this area has been extensive: Cooper [2] was an early contributor. An assortment of applications has been examined: warehouses, audit offices and ambulance centres represent but a few (see Ref. [3]).

Maze et al. [4,5] created a MIP formulation for the transit centre problem. In their model, they attempted to minimize vehicle deadhead as well as driver relief (the costs associated with relieving drivers, when vehicle service schedules extended longer than the maximum driver time stipulated by union contract). They also considered the operating and capital costs of transit facilities. Khumawala’s [6] delta and omega heuristics were used to compare the incremental variable transportation costs against the fixed cost of constructing a transit centre. Maze and his group were able to successfully apply this methodology to a transit system in Detroit, MI.

Waters et al. [7] analysed bus garage location for Calgary, Canada. Their approach considered vehicle deadhead as well as transit centre operating and construction costs. Further, they allowed environmental impact costs by proscribing a penalty cost to any bus deadheading through areas with high residential population densities.

Using an iterative procedure involving a network flow-based algorithm and a location interchange heuristic, Ball et al. [8] developed an approach for examining a relatively large number of possible transit centre locations. Their model was implemented in southeastern Pennsylvania for a system that included 11 current garages, 1400 buses and 800 potential garage locations.

3. Transit centre location model

The VRTS serves the largest transit service area in Canada, covering an area roughly 1800 km². It employs over 4000 individuals, while serving a population base of over 2 million people. Typical weekday ridership is over 375,000, with about 130 million passengers using the system annually. Its system includes over 900 buses (250 electrically powered trolleys and over 650 diesel buses) as well as a ferry system (*SeaBus*) and light rapid transit (*SkyTrain*).

The VRTS currently has five transit centres for buses. BC Transit staff identified five additional locations for transit centres (these are termed “candidate” facilities). Table 1 lists each site, its current capacity and allocation (for existing transit centres), the minimum and maximum capacities used in the mathematical program and whether or not the transit centre can house trolley buses. Sites are indicated with an “E” or “C” to denote existing or candidate transit centres, respectively. Fig. 1 illustrates the location of the ten transit

centres, as well as other prominent transit features in the system.

The MIP model used to determine the optimal number, location and size of transit centres to serve the route network of the VRTS appears as follows (further information regarding such issues as parameter estimation is provided in Willoughby [9]):

MINIMIZE TOTAL COST

$$= \sum_r \sum_d \sum_p \sum_s C_{rdps} X_{rdps} + \sum_{s=\text{candidate}} V_s N_s + \sum_{s=E4} F_s W_s - \sum_{s=\text{existing}} R_s Z_s - \sum_{s=E5} P_s K_s$$

subject to:

$$\sum_s X_{rdps} = D_{rdp} \quad \forall r, d, p, \tag{1}$$

$$\sum_r X_{rdps} - A_s \leq 0 \quad \forall s, \tag{2}$$

$$N_s - \alpha A_s \geq 0 \quad \forall s, \tag{3}$$

$$N_s - \omega_s Y_s \leq 0 \quad \forall s = \text{candidate}, \tag{4}$$

$$N_s - \lambda_s Y_s \geq 0 \quad \forall s = \text{candidate}, \tag{5}$$

$$N_s - \beta_s (1 - Z_s) \leq 0 \quad \forall s = \text{existing}, \tag{6}$$

$$N_s - \gamma_s (1 - Z_s) \geq 0 \quad \forall s = \text{existing}, \tag{7}$$

$$\sum_{r=\text{trolley}} X_{rdp,E4} - \beta_{E4} W_{E4} \leq 0 \quad \forall d, p, \tag{8}$$

$$N_{E5} + K_{E5} = \beta_{E5} (1 - X_{E5}). \tag{9}$$

Variable restrictions:

$$X_{rdps} \geq 0$$

$$A_s, N_s \geq 0 \text{ and integer}$$

$$W_{E4}, Y_s, Z_s = 0, 1$$

$$K_{E5} \geq 0$$

Parameter restrictions:

$$\omega_s, \lambda_s, \beta_s, \gamma_s \geq 0$$

$$\alpha \geq 1$$

The subscripts denote:

- r* route
- d* day (weekday, Saturday, or Sunday/holiday)
- p* service period (AM peak, PM peak, or different types of all-day service)
- s* transit centre

Model variables are:

- X_{rdps} = the number of buses assigned to route *r* on day *d* for service period *p* from transit centre *s*.
- N_s = the total number of buses assigned to transit centre *s* (including spares).

Table 1
Existing and candidate transit centre locations

Site number	Location	Current allocation	Current capacity	Minimum size	Maximum size	Trolley accessible
E1	North Vancouver	78	60	50	60	
E2	Port Coquitlam	114	250	100	250	
E3	Surrey	132	250	100	250	
E4	Burnaby	166	160	100	160	YES
E5	Oakridge	410	350	100	350	YES
C1	British Columbia Rapid Transit Corp. (BCRTC)	—		100	250	
C2	Cloverdale	—		100	250	
C3	Lougheed Park & Ride	—		100	250	
C4	Richmond	—		100	250	
C5	Main & Terminal	—		100	250	YES

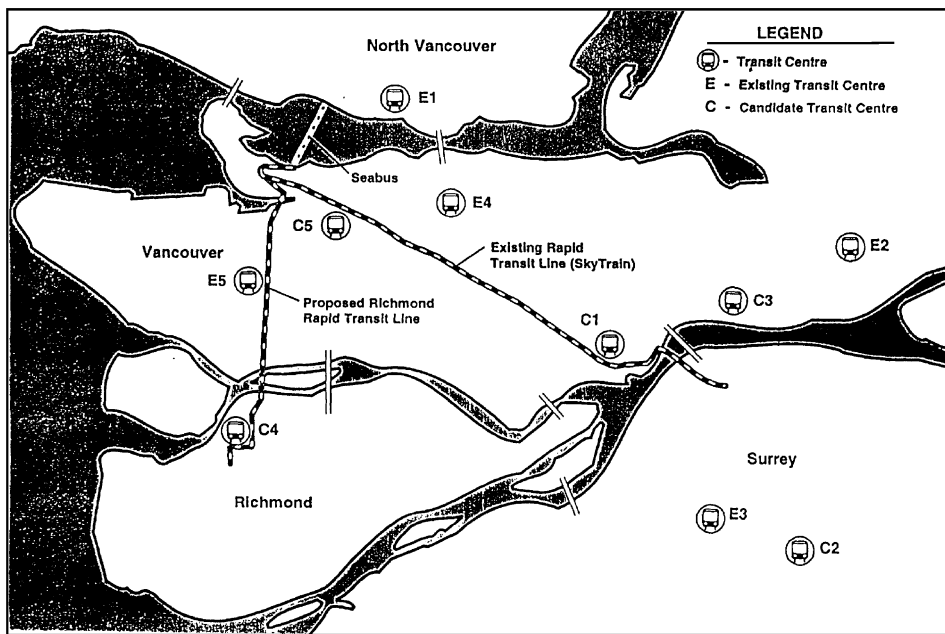


Fig. 1. Location of key transit facilities.

W_{E4} = 0 if Burnaby Transit Centre is not modified to handle trolley buses
 1 if Burnaby Transit Centre is modified to handle trolley buses

(Since this variable and its cost coefficient only apply to one transit centre location, the subscript is given as “E4”, instead of “s”.)

Z_s = 0 if existing transit centre s remains open.
 1 if existing transit centre s is shut down.

A_s = the active number of buses assigned to transit centre s

Y_s = 0 if a transit centre in candidate location s is not opened

K_{E5} = 1 if a transit centre in candidate location s is opened
 amount of “downsizing” performed at the Oakridge Transit Centre (amount of bus capacity eliminated from this garage)

(Since this variable and its cost coefficient only apply to one transit centre location, the subscript is given as “E5”, instead of “s”.)

Model parameters include the following:

C_{rdps} = the annual deadheading cost to operate one bus on route r on day d for service period p from transit centre s .

- V_s = annualized per bus capital costs to construct a candidate transit centre at site s .
- F_{E4} = annualized capital costs to modify the Burnaby Transit Centre so that trolley buses can be assigned to it
- R_s = annualized “salvage value” obtained should existing transit centre s be eliminated.
- P_{E5} = annualized “per bus” salvage value of downsizing the Oakridge Transit Centre
- D_{rdp} = number of buses required (“demand”) for route r on day d during service period p
- α = spare factor used to augment the number of active buses to allow for vehicle breakdowns, major scheduled maintenance, or accidents.
- ω_s = maximum allowed size of candidate transit centre s
- λ_s = minimum allowed size of candidate transit centre s
- β_s = maximum allowed size of existing transit centre s
- γ_s = minimum allowed size of existing transit centre s

The objective function sums the total location/allocation costs. The first term in the objective function comprises the total vehicle deadhead cost. In our model, we assume that a bus is pulled into service at one of the end points of a route and pulled out of service from the other terminus *except* for those routes which offer all-day suburban to downtown Vancouver service. For these routes, transit officials indicated the more realistic scenario involved the bus being both pulled into and removed from service at the suburban terminus.

The second term in the objective function calculates the total capital cost required to construct candidate transit centres. We note that these costs are linear in terms of the number of buses assigned to the centre. We did not include the fixed costs associated with locating a facility at a candidate site, since transit officials were confident that construction costs were linear between the minimum and maximum allowable candidate centre size. Further, any fixed costs were felt to be relatively small in comparison to the variable construction costs.

The third term is used to assign a capital cost should the Burnaby Transit Centre be modified to accept trolley buses. The fourth term considers the salvage value that would accrue to the transit system should an existing transit centre be closed. The final term in the objective function examines the possibility of partially downsizing the Oakridge Transit Centre. Recognizing the tremendous land value of this garage, transit officials wanted the model to consider the impact of eliminating a portion of the depot, should it not be attractive to close the *entire* facility. The “per bus” downsizing value was obtained by dividing the annualized salvage value by the garage’s current capacity.

The first set of constraints ensures that the total demand for buses (in terms of specific route-day-service period combinations) is satisfied. The second type of constraints sums the total number of buses assigned to a specific transit centre and ensures that it does not exceed A_s , the active

number of buses allocated to a transit centre. Since a bus operating in the a.m. peak can also operate in the p.m. peak, a garage’s active number of buses is the maximum of either the all-day plus a.m. peak runs, or the all-day plus p.m. peak runs. We note that this summation is separately done over the three different day periods considered (weekday, Saturday, and Sunday/holiday). Moreover, since weekday bus requirements tend to dominate those of the weekends or holidays, we observe that the active number of buses assigned to a garage is almost always determined by weekday bus assignments.

The next set of constraints augments the active number of buses by a spare factor, α , to give the total number of buses assigned to a transit centre, N_s . The sizes of the candidate transit centres were forced to fall within pre-assigned capacity bounds. This reflected the judgments of transit staff concerning the appropriate values for minimum and maximum sizes of transit centres. The fourth and fifth types of constraints exhibit this. The next two categories of constraints perform a similar function for existing transit centres.

The eighth constraint set ensures that, if trolleys are assigned to the Burnaby Transit Centre (and the concomitant capital charge is incurred), the total allocation cannot exceed the transit centre’s capacity. The final constraint suggests that if the Oakridge garage remains open ($Z_{E5} = 0$), then the total number of buses assigned to the garage plus any downsizing must equal the garage’s current capacity. One cannot downsize by more buses than the current capacity. Should the Oakridge garage be closed ($Z_{E5} = 1$), then the constraint would not permit any buses to be allocated to the facility. Likewise, partial downsizing could not occur (since the entire facility would have been shut down).

The mathematical program consisted of 12,192 continuous assignment variables (X_{rdps}). These are determined in the following manner: 98 diesel routes \times 3 days \times 4 service periods \times 10 locations gives 11,760 variables. Similarly for the trolley routes, we have 12 routes \times 3 days \times 4 service periods \times 3 locations for 432 variables. There are 32 integer variables (21 general and 11 binary) as well as over 2,000 constraints.

A state-of-the-art commercial mixed integer programming package, *CPLEX* 2.0, was employed to solve this MIP problem. All model runs were performed at The University of British Columbia on a HP 9000 series 700 model 730 workstation. This workstation employed a 66 MHz PA-RISC processor supporting the HP-UX UNIX operating system. Solution times to optimality were in the neighborhood of 5–7 min. Table 2 presents the optimal location and size of transit centres for the VRTS in the original MIP model. The total number of buses allocated in the current and optimal solutions may not necessarily be equivalent due to the “rounding up” that could occur when A_s is augmented by α to produce N_s (note that A_s and N_s must take on integer values). Table 3 illustrates the costs associated with the optimal solution.

Table 2
MIP optimal solution results

Site number	Location	Current allocation	Optimal allocation
E1	North Vancouver	78	60
E2	Port Coquitlam	114	127
E3	Surrey	132	157
E4	Burnaby	166	160
E5	Oakridge	410	200
C1	BCRTC	—	0
C2	Cloverdale	—	0
C3	Lougheed Park & Ride	—	0
C4	Richmond	—	101
C5	Main & Terminal	—	104

Table 3
Breakdown of costs MIP optimal solution

Cost category	Current solution	Optimal solution
Deadhead costs	\$14,528,000	\$12,772,845
Capital costs: Richmond	0	\$963,338
Capital costs: Main & Terminal	0	\$1,212,640
Salvage values: Oakridge	0	(\$1,150,800)
Total costs	\$14,528,000	\$13,798,023

It is apparent that the optimal solution calls for a substantial shift in the transit system's location-allocation scheme. All existing transit centres are retained; however, their capacities are now strictly obeyed. Two new facilities are constructed: one in Richmond, and the other near the Main & Terminal intersection in Vancouver proper. Moreover, the Oakridge Transit Centre's allocation is reduced to 200 buses. Trolley buses are not assigned to the Burnaby Transit Centre.

Annual savings of over \$700,000 are produced by the optimal solution (deadhead costs are reduced by over \$1.7 million). This reflects a drop of 5.02% in overall transit centre costs.

4. Additional transit planning scenarios

The development of a model to examine transit centre costs is important from a mathematical perspective. However, from a managerial perspective, the true value in any modelling exercise is derived from one's ability to examine how solutions change under different planning scenarios. We shall outline the effects of three different scenarios in this section.

Table 4
Allocations under MIP extensions

Site number	Location	No. candidate facilities	Forced Oakridge (350)	Greenfields approach
E1	North Vancouver	60	60	0
E2	Port Coquitlam	140	105	100
E3	Surrey	196	133	100
E4	Burnaby	160	160	104
E5	Oakridge	350	350	156
C1	BCRTC	0	100	100
C2	Cloverdale	0	0	0
C3	Lougheed Park & Ride	0	0	0
C4	Richmond	0	0	100
C5	Main & Terminal	0	0	249

4.1. No candidate facilities

Recall that the optimal solution to the MIP model called for the construction of two new bus garages. Transit officials were especially interested in determining the associated costs if the model did not permit the opening of any transit garages. That is, what annual costs would one expect if the current transit garage scheme was used to house all buses, and no overcrowding was permitted? The resulting allocations for all scenarios described in Section 4 are provided in Table 4.

Changing the MIP model to reflect this situation was straightforward. We simply forced each of the Y_s values to be zero. The annual costs of this solution were roughly \$14.603 million, an increase of about \$75,000 from those costs currently experienced. As a result, if transit officials had to use the current transit garage system to alleviate overcrowding, it would cost them an additional \$75,000 per year.

4.2. Forced Oakridge allocation

The MIP optimal solution suggested the Oakridge garage be reduced to 200 buses. What would occur if the mathematical model forced an allocation of 350 buses (current capacity) to the Oakridge Transit Centre?

Table 4 illustrates that by forcing 350 buses to be assigned to the Oakridge garage, the attractiveness of the Main & Terminal facility (located close to the Oakridge garage) has been diminished. This candidate facility was opened in the original optimal solution, but in this scenario, it is closed. The same situation is observed for the Richmond bus garage. The candidate facility near the BCRTC site is opened.

The total costs of this solution are roughly \$13.985 million, about \$500,000 less than current transit system costs. However, these costs are over \$200,000 more than those found in the MIP optimal solution.

4.3. Greenfields approach

A scenario that especially intrigued transit planners was the following. Suppose that the transit system had a “clean slate” with which to work (a so-called “Greenfields” approach). Transit routes remained as currently established, but any number of the ten potential locations could be chosen. There were no construction costs associated with building a candidate facility, nor salvage values with closing an existing garage (minimum and maximum allocations of 100 and 250, respectively, were used). If transit planners had such a clean slate before them, which garages would one choose and—perhaps more importantly—how many buses would be allocated to each transit centre?

The changes in the transit system depicted in Table 4 are quite interesting. One of the candidate facilities, the Main & Terminal location, received the largest allocation of buses (249). Ignoring construction costs, this garage would appear to be more ideally situated than the Oakridge Transit Centre (this latter facility had a total allocation of 156 buses). One of the existing garages, North Vancouver, does not receive any buses under this approach.

5. Conclusions

A mixed integer programming model has been developed to determine the optimal number, location and size of transit centres for the Vancouver Regional Transit System. It would appear advantageous for transit officials to explore the possibility of constructing new garages at the Main & Terminal and Richmond locations. (Subsequent to preparation of this manuscript, we have learned that a transit centre has been planned for the Richmond area.)

We have explored how this model can be used to examine additional transit planning scenarios. This quantitative tool can assist the decision-making process by quickly

illustrating the cost and allocation effects of these new scenarios.

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